CHAPTER

12

OPERATIONS WITH RADICALS

Whenever a satellite is sent into space, or astronauts are sent to the moon, technicians at earthbound space centers monitor activities closely. They continually make small corrections to help the spacecraft stay on course.

The distance from the earth to the moon varies, from 221,460 miles to 252,700 miles, and both the earth and the moon are constantly rotating in space. A tiny error can send the craft thousands of miles off course. Why do such errors occur?

Space centers rely heavily on sophisticated computers, but computers and calculators alike work with *approximations* of numbers, not necessarily with *exact* values.

We have learned that irrational numbers, such as $\sqrt{2}$ and $\sqrt{5}$, are shown on a calculator as decimal approximations of their true values. All irrational numbers, which include radicals, are nonrepeating decimals that never end. How can we work with them?

In this chapter, we will learn techniques to compute with radicals to find exact answers. We will also look at methods for working with radicals on a calculator to understand how to minimize errors when using these devices.

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12-1 RADICALS AND THE RATIONAL NUMBERS

Squares and Square Roots

Recall from Section 1-3 that to *square* a number means to multiply the number by itself. To square 8, we write:

$$8^2 = 8 \times 8 = 64$$

On a calculator:

ENTER: 8 $\boxed{x^2}$ $\boxed{\text{ENTER}}$

DISPLAY:

```
8²
                64
```

To find the *square root* of a number means to find the value that, when multiplied by itself, is equal to the given number. To express the square root of 64, we write:

$$\sqrt{64} = 8$$

On a calculator:

ENTER: $\boxed{\text{2nd}}$ $\boxed{\sqrt{\ }}$ 64 $\boxed{\text{ENTER}}$

DISPLAY:

```
√(64
                 8
```

The symbol $\sqrt{\ }$ is called the *radical sign*, and the quantity under the radical sign is the **radicand**. For example, in $\sqrt{64}$, which we read as "the square root of 64," the radicand is 64.

A **radical**, which is any term containing both a radical sign and a radicand, is a root of a quantity. For example, $\sqrt{64}$ is a radical.

► **In general, the square root of b is x (written as $\sqrt{b} = x$) if and only if $x \geq 0$ and $x^2 = b$.**

Some radicals, such as $\sqrt{4}$ and $\sqrt{9}$, are rational numbers; others, such as $\sqrt{2}$ and $\sqrt{3}$, are irrational numbers. We begin this study of radicals by examining radicals that are rational numbers.

Perfect Squares

Any number that is the square of a rational number is called a **perfect square**. For example,

$$3 \times 3 = 9 \qquad 0 \times 0 = 0 \qquad 1.4 \times 1.4 = 1.96 \qquad \tfrac{2}{7} \times \tfrac{2}{7} = \tfrac{4}{49}$$

Therefore, perfect squares include 9, 0, 1.96, and $\frac{4}{49}$.

Then, by applying the inverse operation, we know that:

▶ **The square root of every perfect square is a *rational* number.**

$$\sqrt{9} = 3 \qquad \sqrt{0} = 0 \qquad \sqrt{1.96} = 1.4 \qquad \sqrt{\tfrac{4}{49}} = \tfrac{2}{7}$$

Radicals That Are Square Roots

Certain generalizations can be made for *all* radicals that are square roots, whether they are rational numbers or irrational numbers:

1. Since the square root of 36 is a number whose square is 36, we can write the statement $\left(\sqrt{36}\right)^2 = 36$. We notice that $\left(\sqrt{36}\right)^2 = (6)^2 = 36$. It is also true that $\sqrt{(6)^2} = \sqrt{36} = 6$.

▶ **In general, for every nonnegative real number n: $\left(\sqrt{n}\right)^2 = n$ and $\sqrt{n^2} = n$.**

2. Since $(+6)(+6) = 36$ and $(-6)(-6) = 36$, both $+6$ and -6 are square roots of 36. This example illustrates the following statement:

▶ **Every positive number has two square roots: one root is a positive number called the *principal square root*, and the other root is a negative number. These two roots have the same absolute value.**

To indicate the positive or principal square root only, place a radical sign over the number:

$$\sqrt{25} = 5 \qquad \text{and} \qquad \sqrt{0.49} = 0.7$$

To indicate the negative square root only, place a negative sign before the radical:

$$-\sqrt{25} = -5 \qquad \text{and} \qquad -\sqrt{0.49} = -0.7$$

To indicate both square roots, place both a positive and a negative sign before the radical:

$$\pm\sqrt{25} = \pm 5 \qquad \text{and} \qquad \pm\sqrt{0.49} = \pm 0.7$$

3. Every real number, when squared, is either positive or 0. Therefore:

▶ **The square root of a negative number does not exist in the set of real numbers.**

For example, $\sqrt{-25}$ does not exist in the set of real numbers because there is no real number that, when multiplied by itself, equals -25.

Calculators and Square Roots

A calculator will return only the principal square root of a positive number. To display the negative square root of a number, the negative sign must be entered before the radical.

ENTER: **(-)** **2nd** **√** 25 **ENTER**

DISPLAY:

```
-√(25
                -5
```

The calculator will display an error message if it is set in "real" mode and the square root of a negative number is entered.

ENTER: **2nd** **√** **(-)** 25 **ENTER**

DISPLAY:

```
ERR:NONREAL ANS
1:QUIT
2:GOTO
```

Cube Roots and Other Roots

A **cube root** of a number is one of the three equal factors of the number. For example, 2 is a cube root of 8 because $2 \times 2 \times 2 = 8$, or $2^3 = 8$. A cube root of 8 is written as $\sqrt[3]{8}$.

▶ **Finding a cube root of a number is the inverse operation of cubing a number. In general, the cube root of b is x (written as $\sqrt[3]{b} = x$) if and only if $x^3 = b$.**

We have said that $\sqrt{-25}$ does not exist in the set of real numbers. However, $\sqrt[3]{-8}$ does exist in the set of real numbers. Since $(-2)^3 = (-2)(-2)(-2) = -8$, then $\sqrt[3]{-8} = -2$.

In the set of real numbers, every number has one cube root. The cube root of a positive number is positive, and the cube root of a negative number is negative.

In the expression $\sqrt[n]{b}$, the integer n that indicates the root to be taken, is called the **index** of the radical. Here are two examples:

- In $\sqrt[3]{8}$, read as "the cube root of 8," the index is 3.
- In $\sqrt[4]{16}$, read as "the fourth root of 16," the index is 4. Since $2^4 = 16$, 2 is one of the four equal factors of 16, and $\sqrt[4]{16} = 2$.

When no index appears, the index is understood to be 2. Thus, $\sqrt{25} = \sqrt[2]{25} = 5$.

When the index of the root is even and the radicand is positive, the value is a real number. That real number is positive if a plus sign or no sign precedes the radical, and negative if a minus sign precedes the radical.

When the index of the root is even and the radicand is negative, the root has no real number value.

When the index of the root is odd and the radicand either positive or negative, the value is a real number. That real number is positive if the radicand is positive and negative if the radicand is negative.

Calculators and Roots

A radical that has an index of 3 or larger can be evaluated on most graphing calculators. To do so, first press the MATH key to display a list of choices. Cube root is choice 4. To show that $\sqrt[3]{64} = 4$:

ENTER: MATH 4 64 ENTER

DISPLAY:
```
³√(64
                    4
```

Any root with an index greater than 3 can be found using choice 5 of the MATH menu. The index, indicated by x in the menu, must be entered first.

To show that $\sqrt[4]{625} = 5$:

ENTER: 4 MATH 5 625 ENTER

DISPLAY:
```
4ˣ√625
                    5
```

EXAMPLE 1

Find the principal square root of 361.

Solution Since $19 \times 19 = 361$, then $\sqrt{361} = 19$.

Calculator Solution ENTER: 2nd $\sqrt{\ }$ 361 ENTER

DISPLAY:

```
√(361
                19
```

Answer 19

EXAMPLE 2

Find the value of $-\sqrt{0.0016}$.

Solution Since $(0.04)(0.04) = 0.0016$, then $-\sqrt{0.0016} = -0.04$.

Calculator Solution ENTER: (-) 2nd $\sqrt{\ }$ .0016 ENTER

DISPLAY:

```
-√(.0016
               -.04
```

Answer -0.04

EXAMPLE 3

Is $\left(\sqrt{13}\right)^2$ a rational or an irrational number? Explain your answer.

Solution Since $\left(\sqrt{n}\right)^2 = n$ for $n \geq 0$, then $\left(\sqrt{13}\right)^2 = 13$.

Answer The quantity $\left(\sqrt{13}\right)^2$ is a rational number since its value, 13, can be written as the quotient of two integers, $\frac{13}{1}$, where the denominator is not 0.

Note: $\left(\sqrt{-13}\right)^2$ does not exist in the set of real numbers since, by the order of operations, $\sqrt{-13}$ must be evaluated first. There is no real number that, when squared, equals -13.

EXAMPLE 4

Solve for x: $x^2 = 36$.

Solution If $x^2 = a$, then $x = \pm\sqrt{a}$ when a is a positive number.

	Check	
$x^2 = 36$	$x^2 = 36$	$x^2 = 36$
$x = \pm\sqrt{36}$	$(+6)^2 = 36$	$(-6)^2 = 36$
$x = \pm 6$	$36 = 36$ ✔	$36 = 36$ ✔

Answer $x = +6$ or $x = -6$; the solution set is $\{+6, -6\}$.

EXERCISES

Writing About Mathematics

1. Explain the difference between $-\sqrt{9}$ and $\sqrt{-9}$.

2. We know that $5^3 = 125$ and $5^4 = 625$. Explain why $\sqrt[3]{-125} = -5$ but $\sqrt[4]{-625} \neq -5$.

Developing Skills

In 3–22, express each radical as a rational number with the appropriate signs.

3. $\sqrt{16}$ **4.** $-\sqrt{64}$ **5.** $\pm\sqrt{100}$ **6.** $\pm\sqrt{169}$ **7.** $\sqrt{400}$

8. $-\sqrt{625}$ **9.** $\sqrt{\frac{1}{4}}$ **10.** $-\sqrt{\frac{9}{16}}$ **11.** $\pm\sqrt{\frac{25}{81}}$ **12.** $\sqrt{0.64}$

13. $-\sqrt{1.44}$ **14.** $\pm\sqrt{0.09}$ **15.** $\pm\sqrt{0.0004}$ **16.** $\sqrt[3]{1}$ **17.** $\sqrt[4]{81}$

18. $\sqrt[5]{32}$ **19.** $\sqrt[3]{-8}$ **20.** $-\sqrt[3]{-125}$ **21.** $\sqrt[4]{0.1296}$ **22.** $-\sqrt{\frac{36}{4}}$

In 23–32, evaluate each radical by using a calculator.

23. $\sqrt{10.24}$ **24.** $-\sqrt{46.24}$ **25.** $\sqrt[3]{2.197}$ **26.** $\sqrt[3]{-3{,}375}$ **27.** $\sqrt[4]{4{,}096}$

28. $\sqrt[5]{-1{,}024}$ **29.** $-\sqrt[3]{-1{,}000}$ **30.** $-\sqrt{32.49}$ **31.** $\sqrt[3]{-0.125}$ **32.** $\pm\sqrt{5.76}$

In 33–38, find the value of each expression in simplest form.

33. $\left(\sqrt{8}\right)^2$ **34.** $\sqrt{\left(\frac{1}{2}\right)^2}$ **35.** $\left(\sqrt{0.7}\right)^2$

36. $\sqrt{\left(\frac{9}{3}\right)^2}$ **37.** $\left(\sqrt{97}\right)\left(\sqrt{97}\right)$ **38.** $\sqrt{(-9)^2} + \left(\sqrt{83}\right)^2$

In 39–47, replace each ? with >, <, or = to make a true statement.

39. $\frac{3}{4}$? $\left(\frac{3}{4}\right)^2$

40. 1 ? 1^2

41. $\frac{3}{2}$? $\left(\frac{3}{2}\right)^2$

42. $\frac{1}{9}$? $\sqrt{\frac{1}{9}}$

43. $\frac{4}{25}$? $\sqrt{\frac{4}{25}}$

44. 1 ? $\sqrt{1}$

45. m ? $\sqrt{m}, 0 < m < 1$

46. m ? $\sqrt{m}, m = 1$

47. m ? $\sqrt{m}, m > 1$

In 48–55, solve each equation for the variable when the replacement set is the set of real numbers.

48. $x^2 = 4$

49. $y^2 = \frac{4}{81}$

50. $x^2 = 0.49$

51. $x^2 - 16 = 0$

52. $y^2 - 30 = 6$

53. $2x^2 = 50$

54. $3y^2 - 27 = 0$

55. $x^3 = 8$

Applying Skills

In 56–59, in each case, find the length of the hypotenuse of a right triangle when the lengths of the legs have the given values.

56. 6 inches and 8 inches

57. 5 centimeters and 12 centimeters

58. 15 meters and 20 meters

59. 15 feet and 36 feet

In 60–63, find, in each case: **a.** the length of each side of a square that has the given area **b.** the perimeter of the square.

60. 36 square feet

61. 196 square yards

62. 121 square centimeters

63. 225 square meters

64. Express in terms of x, $(x > 0)$, the perimeter of a square whose area is represented by x^2.

65. Write each of the integers from 101 to 110 as the sum of the smallest possible number of perfect squares. (For example, $99 = 7^2 + 7^2 + 1^2$.) Use positive integers only.

12-2 RADICALS AND THE IRRATIONAL NUMBERS

We have learned that $\sqrt{n}$ is a rational number when n is a perfect square. What type of number is $\sqrt{n}$ when n is *not* a perfect square? As an example, let us examine $\sqrt{5}$ using a calculator.

ENTER: **2nd** **√** 5 **ENTER**

DISPLAY:

```
√(5
          2.236067977
```

Can we state that $\sqrt{5} = 2.236067977$? Or is 2.236067977 a rational approximation of $\sqrt{5}$? To answer this question, we will find the square of 2.236067977.

ENTER: 2.236067977 x^2 ENTER

DISPLAY:

```
2.236067977²
         4.999999998
```

We know that if $\sqrt{b} = x$, then $x^2 = b$. The calculator displays shown above demonstrate that $\sqrt{5} \neq 2.236067977$ because $(2.236067977)^2 \neq 5$. The rational number 2.236067977 is an approximate value for $\sqrt{5}$.

Recall that a number such as $\sqrt{5}$ is called an *irrational number*. Irrational numbers cannot be expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Furthermore, irrational numbers cannot be expressed as terminating or repeating decimals. The example above illustrates the truth of the following statement:

► **If n is any positive number that is *not* a perfect square, then $\sqrt{n}$ is an irrational number.**

Radicals and Estimation

The radical $\sqrt{5}$ represents the *exact value* of the irrational number whose square is 5. The calculator display for $\sqrt{5}$ is a *rational approximation* of the irrational number. It is a number that is close to, but not equal to $\sqrt{5}$. There are other values correctly rounded from the calculator display that are also approximations of $\sqrt{5}$:

2.236067977	calculator display, to nine decimal places
2.236068	rounded to six decimal places (nearest millionth)
2.23607	rounded to five decimal places (nearest hundred-thousandth)
2.2361	rounded to four decimal places (nearest ten-thousandth)
2.236	rounded to three decimal places (nearest thousandth)
2.24	rounded to two decimal places (nearest hundredth)

Each rational approximation of $\sqrt{5}$, as seen above, indicates that $\sqrt{5}$ is greater than 2 but less than 3. This fact can be further demonstrated by placing the square of $\sqrt{5}$, which is 5, between the squares of two consecutive integers, one less than 5 and one greater than 5, and then finding the square root of each number.

$$\begin{array}{lrcrcr} \text{Since} & 4 & < & 5 & < & 9, \\ \text{then} & \sqrt{4} & < & \sqrt{5} & < & \sqrt{9}, \\ \text{or} & 2 & < & \sqrt{5} & < & 3. \end{array}$$

In the same way, to get a quick estimate of any square-root radical, we place its square between the squares of two consecutive integers. Then we take the square root of each term to show between which two consecutive integers the radical lies. For example, to estimate $\sqrt{73}$:

$$\begin{array}{llllll} \text{Since} & 64 & < & 73 & < & 81, \\ \text{then} & \sqrt{64} & < & \sqrt{73} & < & \sqrt{81}, \\ \text{or} & 8 & < & \sqrt{73} & < & 9. \end{array}$$

Basic Rules for Radicals That Are Irrational Numbers

There are general rules to follow when working with radicals, especially those that are irrational numbers:

1. If the degree of accuracy is not specified in a question, it is best to give the *exact* answer in radical form. In other words, if the answer involves a radical, leave the answer in radical form.

For example, the sum of 2 and $\sqrt{5}$ is written as $2 + \sqrt{5}$, an *exact* value.

2. If the degree of accuracy is not specified in a question and a rational approximation is to be given, the approximation should be correct to *two or more* decimal places.

For example, an exact answer is $2 + \sqrt{5}$. By using a calculator, a student discovers that $2 + \sqrt{5}$ is approximately $2 + 2.236067977 = 4.236067977$. *Acceptable answers* would include the calculator display and correctly rounded approximations of the display to two or more decimal places:

4.2360680	(seven places)	4.2361	(four places)
4.236068	(six places)	4.236	(three places)
4.23607	(five places)	4.24	(two places)

Unacceptable answers would include values that are not rounded correctly, as well as values with fewer than two decimal places such as 4.2 (one decimal place) and 4 (no decimal place).

3. When the solution to a problem involving radicals has two or more steps, no radical should be rounded until the very last step.

For example, to find the value of $3 \times \sqrt{5}$, rounded to the *nearest hundredth*, first multiply the calculator approximation for $\sqrt{5}$ by 3 and then round the product to two decimal places.

Correct Solution: $3\left(\sqrt{5}\right) = 3(2.236067977) = 6.708203931 \approx 6.71$

Incorrect Solution: $3\left(\sqrt{5}\right) = 3(2.236067977) \approx 3(2.24) = 6.72$

The solution, 6.72, is incorrect because the rational approximation of $\sqrt{5}$ was rounded too soon. To the nearest hundredth, the correct answer is 6.71.

EXAMPLE 1

Between which two consecutive integers is $-\sqrt{42}$?

Solution

How to Proceed	
(1) Place 42 between the squares of consecutive integers:	$36 < 42 < 49$
(2) Take the square root of each number:	$\sqrt{36} < \sqrt{42} < \sqrt{49}$
(3) Simplify terms:	$6 < \sqrt{42} < 7$
(4) Multiply each term of the inequality by -1: Recall that when an inequality is multiplied by a negative number, the order of the inequality is reversed.	$-6 < -\sqrt{42} < -7$ $-7 > -\sqrt{42} > -6$

Answer $-\sqrt{42}$ is between -7 and -6.

EXAMPLE 2

Is $\sqrt{56}$ a rational or an irrational number?

Solution Since 56 is a positive integer that is not a perfect square, there is no rational number that, when squared, equals 56. Therefore, $\sqrt{56}$ is irrational.

Calculator Solution

STEP 1. Evaluate $\sqrt{56}$.

ENTER: **2nd** **$\sqrt{\ }$** 56 **ENTER**

DISPLAY:
```
√(56
          7.483314774
```

STEP 2. To show that the calculator displays a rational approximation, not an exact value, show that the square of 7.483314774 does not equal 56.

ENTER: 7.483314774 **x^2** **ENTER**

DISPLAY:
```
7.483314774²
         56.00000001
```

Since $(7.483314774)^2 \neq 56$, then 7.483314774 is a rational approximation for $\sqrt{56}$, not an exact value.

Answer $\sqrt{56}$ is an irrational number.

EXAMPLE 3

Is $\sqrt{8.0656}$ rational or irrational?

Calculator Solution **STEP 1.** Evaluate $\sqrt{8.0656}$.

ENTER: 2nd $\sqrt{\ }$ 8.0656 ENTER

DISPLAY:

```
√(8.0656
                2.84
```

STEP 2. It appears that the rational number in the calculator display is an exact value of $\sqrt{8.0656}$. To verify this, show that the square of 2.84 does equal 8.0656.

ENTER: 2.84 x^2 ENTER

DISPLAY:

```
2.84²
              8.0656
```

Since $(2.84)^2 = 8.0656$, then $\sqrt{8.0656} = 2.84$.

Answer $\sqrt{8.0656}$ is a rational number.

If n is a positive rational number written as a terminating decimal, then n^2 has twice as many decimal places as n. For example, the square of 2.84 has four decimal places. Also, since the last digit of 2.84 is 4, note that the last digit of 2.84^2 must be 6 because $4^2 = 16$.

EXAMPLE 4

Approximate the value of the expression $\sqrt{8^2 + 13} - 4$

a. to the *nearest thousandth* **b.** to the *nearest hundredth*

Calculator Solution ENTER: 2nd $\sqrt{\ }$ 8 x^2 + 13) − 4 ENTER

DISPLAY:

```
√(8²+13)-4
          4.774964387
```

a. To round to the *nearest thousandth*, look at the digit in the ten-thousandth (4th) decimal place. Since this digit (9) is greater than 5, add 1 to the digit in the thousandth (3rd) decimal place. When rounded to the nearest thousandth, 4.774964387 is approximately equal to 4.775

b. To round to the *nearest hundredth*, look at the digit in the thousandth (3rd) decimal place. Since this digit (4) is not greater than or equal to 5, drop this digit and all digits to the right. When rounded to the nearest hundredth, 4.774964387 is approximately equal to 4.77

Answers **a.** 4.775 **b.** 4.77

Note: It is not correct to round 4.774964387 to the nearest hundredth by rounding 4.775, the approximation to the nearest thousandth.

EXERCISES

Writing About Mathematics

1. a. Use a calculator to evaluate $\sqrt{999{,}999}$.

b. Enter your answer to part **a** and square the answer. Is the result 999,999?

c. Is $\sqrt{999{,}999}$ rational or irrational? Explain your answer.

2. Ursuline said that $\sqrt{\frac{18}{50}}$ is an irrational number because it is the ratio of $\sqrt{18}$ which is irrational and $\sqrt{50}$ which is irrational. Do you agree with Ursuline? Explain why or why not.

Developing Skills

In 3–12, between which consecutive integers is each given number?

3. $\sqrt{8}$ **4.** $\sqrt{13}$ **5.** $\sqrt{40}$ **6.** $-\sqrt{2}$ **7.** $-\sqrt{14}$

8. $\sqrt{52}$ **9.** $\sqrt{73}$ **10.** $-\sqrt{125}$ **11.** $\sqrt{143}$ **12.** $\sqrt{9 + 36}$

In 13–18, in each case, write the given numbers in order, starting with the smallest.

13. $2, \sqrt{3}, -1$ **14.** $4, \sqrt{17}, 3$ **15.** $-\sqrt{15}, -3, -4$

16. $0, \sqrt{7}, -\sqrt{7}$ **17.** $5, \sqrt{21}, \sqrt{30}$ **18.** $-\sqrt{11}, -\sqrt{23}, -\sqrt{19}$

In 19–33, state whether each number is rational or irrational.

19. $\sqrt{25}$ **20.** $\sqrt{40}$ **21.** $-\sqrt{36}$ **22.** $-\sqrt{54}$ **23.** $-\sqrt{150}$

24. $\sqrt{400}$ **25.** $\sqrt{\frac{1}{2}}$ **26.** $-\sqrt{\frac{4}{9}}$ **27.** $\sqrt{0.36}$ **28.** $\sqrt{0.1}$

29. $\sqrt{1{,}156}$ **30.** $\sqrt{951}$ **31.** $\sqrt{6.1504}$ **32.** $\sqrt{2{,}672.89}$ **33.** $\sqrt{5.8044}$

In 34–48, for each irrational number, write a rational approximation: **a.** as shown on a calculator display **b.** rounded to four decimal places

34. $\sqrt{2}$ **35.** $\sqrt{3}$ **36.** $\sqrt{21}$ **37.** $\sqrt{39}$ **38.** $\sqrt{80}$

39. $\sqrt{90}$ **40.** $\sqrt{108}$ **41.** $\sqrt{23.5}$ **42.** $\sqrt{88.2}$ **43.** $-\sqrt{115.2}$

44. $\sqrt{28.56}$ **45.** $\sqrt{67.25}$ **46.** $\sqrt{4{,}389}$ **47.** $\sqrt{123.7}$ **48.** $\sqrt{134.53}$

In 49–56, use a calculator to approximate each expression: **a.** to the *nearest hundredth* **b.** to the *nearest thousandth*.

49. $\sqrt{7} + 7$ **50.** $\sqrt{2} + \sqrt{6}$ **51.** $\sqrt{8} + \sqrt{8}$ **52.** $\sqrt{50} + 17$

53. $2 + \sqrt{3}$ **54.** $19 - \sqrt{3}$ **55.** $\sqrt{130} - 4$ **56.** $\sqrt{27} + 4.0038$

57. Both $\sqrt{58}$ and $\sqrt{58.01}$ are irrational numbers. Find a rational number n such that $\sqrt{58} < n < \sqrt{58.01}$.

Applying Skills

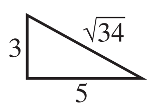
In 58–63, in each case, find to the *nearest tenth* of a centimeter the length of a side of a square whose area is the given measure.

58. 18 square centimeters **59.** 29 square centimeters **60.** 96 square centimeters

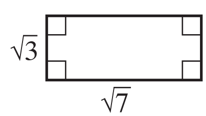
61. 140 square centimeters **62.** 202 square centimeters **63.** 288 square centimeters

In 64–67, find the perimeter of each figure, rounded to the *nearest hundredth*.

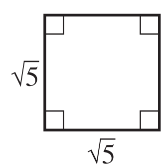
64.



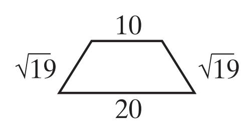
65.



66.



67.



12-3 FINDING THE PRINCIPAL SQUARE ROOT OF A MONOMIAL

Just as we can find the principal square root of a number that is a perfect square, we can find the principal square roots of variables and monomials that represent perfect squares.

- Since $(6)(6) = 36$, then $\sqrt{36} = 6$.
- Since $(x)(x) = x^2$, then $\sqrt{x^2} = x$ $(x \geq 0)$.
- Since $(a^2)(a^2) = a^4$, then $\sqrt{a^4} = a^2$.
- Since $(6a^2)(6a^2) = 36a^4$, then $\sqrt{36a^4} = 6a^2$.

In the last case, where the square root contains both numerical and variable factors, we can determine the square root by finding the square roots of its factors and multiplying:

$$\sqrt{36a^4} = \sqrt{36} \cdot \sqrt{a^4} = 6 \cdot a^2 = 6a^2$$

Procedure

To find the square root of a monomial that has two or more factors, write the indicated product of the square roots of its factors.

Note: In our work, we limit the domain of all variables under a radical sign to *nonnegative* numbers.

EXAMPLE 1

Find the principal square root of each monomial. Assume that all variables represent positive numbers.

a. $25y^2$ **b.** $1.44a^6b^2$ **c.** $1{,}369m^{10}$ **d.** $\frac{81}{4}g^6$

Solution **a.** $\sqrt{25y^2} = \left(\sqrt{25}\right)\left(\sqrt{y^2}\right) = (5)(y) = 5y$ *Answer*

b. $\sqrt{1.44a^6b^2} = \left(\sqrt{1.44}\right)\left(\sqrt{a^6}\right)\left(\sqrt{b^2}\right) = (1.2)(a^3)(b) = 1.2a^3b$ *Answer*

c. $\sqrt{1{,}369m^{10}} = \left(\sqrt{1{,}369}\right)\left(\sqrt{m^{10}}\right) = (37)(m^5) = 37m^5$ *Answer*

d. $\sqrt{\frac{81}{4}g^6} = \left(\sqrt{\frac{81}{4}}\right)(g^6) = \left(\frac{9}{2}\right)(g^3) = \frac{9}{2}g^3$ *Answer*

EXERCISES

Writing About Mathematics

1. Is it true that for $x < 0$, $\sqrt{x^2} = -x$? Explain your answer.
2. Melanie said that when a is an even integer and $x \geq 0$, $\sqrt{x^a} = x^{\frac{a}{2}}$. Do you agree with Melanie? Explain why or why not.

Developing Skills

In 3–14, find the principal square root of each monomial. Assume that all variables represent positive numbers.

3. $4a^2$ **4.** $49z^2$ **5.** $\frac{16}{25}r^2$ **6.** $0.81w^2$

7. $9c^2$ **8.** $36y^4$ **9.** c^2d^2 **10.** $4x^2y^2$

11. $144a^4b^2$ **12.** $0.36m^2$ **13.** $0.49a^2b^2$ **14.** $70.56b^2x^{10}$

Applying Skills

In 15–18, where all variables represent positive numbers:

a. Represent each side of the square whose area is given.

b. Represent the perimeter of that square.

15. $49c^2$ **16.** $64x^2$ **17.** $100x^2y^2$ **18.** $144a^2b^2$

19. The length of the legs of a right triangle are represented by $9x$ and $40x$. Represent the length of the hypotenuse of the right triangle.

12-4 SIMPLIFYING A SQUARE-ROOT RADICAL

A radical that is an irrational number, such as $\sqrt{12}$, can often be simplified. To understand this procedure, let us first consider some radicals that are rational numbers. We know that $\sqrt{36} = 6$ and that $\sqrt{400} = 20$.

Since $\sqrt{4 \cdot 9} = \sqrt{36} = 6$
and $\sqrt{4} \cdot \sqrt{9} = 2 \cdot 3 = 6$,
then $\sqrt{4 \cdot 9} = \sqrt{4} \cdot \sqrt{9}$.

Since $\sqrt{16 \cdot 25} = \sqrt{400} = 20$.
and $\sqrt{16} \cdot \sqrt{25} = 4 \cdot 5 = 20$,
then $\sqrt{16 \cdot 25} = \sqrt{16} \cdot \sqrt{25}$.

These examples illustrate the following property of square-root radicals:

► **The square root of a product of nonnegative numbers is equal to the product of the square roots of these numbers.**

In general, if a and b are nonnegative numbers:

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b} \quad \text{and} \quad \sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$$

Now we will apply this rule to a square-root radical with a radicand that is not a perfect square.

1. Express 50 as the product of 25 and 2, where 25 is the *greatest perfect-square* factor of 50: $\sqrt{50} = \sqrt{25 \cdot 2}$

2. Write the product of the two square roots: $= \sqrt{25} \cdot \sqrt{2}$

3. Replace $\sqrt{25}$ with 5 to obtain the simplified expression with the smallest possible radicand: $= 5\sqrt{2}$

The expression $5\sqrt{2}$ is called the *simplest form* of $\sqrt{50}$. **The simplest form of a square-root radical** is one in which the radicand is an integer that has no perfect-square factor other than 1.

If the radicand is a fraction, change it to an equivalent fraction that has a denominator that is a perfect square. Write the radicand as the product of a fraction that is a perfect square and an integer that has no perfect-square factor other than 1. For example:

$$\sqrt{\tfrac{8}{3}} = \sqrt{\tfrac{8}{3} \times \tfrac{3}{3}} = \sqrt{\tfrac{24}{9}} = \sqrt{\tfrac{4}{9} \times 6} = \sqrt{\tfrac{4}{9}} \times \sqrt{6} = \tfrac{2}{3}\sqrt{6}$$

Procedure

To simplify the square root of a product:

1. If the radicand is a fraction, write it as an equivalent fraction with a denominator that is a perfect square.
2. Find, if possible, two factors of the radicand such that one of the factors is a perfect square and the other is an integer that has no factor that is a perfect square.
3. Express the square root of the product as the product of the square roots of the factors.
4. Find the square root of the factor with the perfect-square radicand.

EXAMPLE 1

Simplify each expression. Assume that $y > 0$.

Answers

a. $\sqrt{18}$ $= \sqrt{9 \cdot 2} = \sqrt{9} \cdot \sqrt{2} = 3\sqrt{2}$

b. $4\sqrt{150}$ $= 4\sqrt{25 \cdot 6} = 4\sqrt{25} \cdot \sqrt{6} = 4 \cdot 5\sqrt{6} = 20\sqrt{6}$

c. $\sqrt{\tfrac{9}{2}}$ $= \sqrt{\tfrac{9}{2}} = \sqrt{\tfrac{9}{2} \times \tfrac{2}{2}} = \sqrt{\tfrac{18}{4}} = \sqrt{\tfrac{9}{4}} \times \sqrt{2} = \tfrac{3}{2}\sqrt{2}$

d. $\sqrt{4y^3}$ $= \sqrt{4y^3} = \sqrt{4y^2 \cdot y} = \sqrt{4y^2} \cdot \sqrt{y} = 2y\sqrt{y}$

EXERCISES

Writing About Mathematics

1. Does $\frac{1}{3}\sqrt{27} = \sqrt{9}$? Explain why or why not.

2. Abba simplified the expression $\sqrt{192}$ by writing $\sqrt{192} = \sqrt{16 \cdot 12} = 4\sqrt{12}$.

a. Explain why $4\sqrt{12}$ is *not* the simplest form of $\sqrt{192}$.

b. Show how it is possible to find the simplest form of $\sqrt{192}$ by starting with $4\sqrt{12}$.

c. What is the simplest form of $\sqrt{192}$?

Developing Skills

In 3–22, write each expression in simplest form. Assume that all variables represent positive numbers.

3. $\sqrt{12}$ **4.** $\sqrt{27}$ **5.** $\sqrt{63}$ **6.** $\sqrt{98}$ **7.** $4\sqrt{12}$

8. $2\sqrt{20}$ **9.** $5\sqrt{24}$ **10.** $\frac{1}{4}\sqrt{48}$ **11.** $\frac{3}{4}\sqrt{96}$ **12.** $\frac{3}{2}\sqrt{80}$

13. $\sqrt{\frac{1}{2}}$ **14.** $\sqrt{\frac{36}{5}}$ **15.** $\sqrt{\frac{32}{3}}$ **16.** $2\sqrt{\frac{1}{8}}$ **17.** $15\sqrt{\frac{2}{5}}$

18. $8\sqrt{9x}$ **19.** $\sqrt{3x^3}$ **20.** $\sqrt{49x^5}$ **21.** $\sqrt{36r^2s}$ **22.** $\sqrt{243xy^2}$

23. The expression $\sqrt{48}$ is equivalent to

(1) $2\sqrt{3}$ (2) $4\sqrt{12}$ (3) $4\sqrt{3}$ (4) $16\sqrt{3}$

24. The expression $4\sqrt{2}$ is equivalent to

(1) $\sqrt{8}$ (2) $\sqrt{42}$ (3) $\sqrt{32}$ (4) $\sqrt{64}$

25. The expression $3\sqrt{18}$ is equivalent to

(1) $\sqrt{54}$ (2) $3\sqrt{2}$ (3) $9\sqrt{2}$ (4) $3\sqrt{6}$

26. The expression $3\sqrt{3}$ is equivalent to

(1) $\sqrt{9}$ (2) $\sqrt{6}$ (3) $\sqrt{12}$ (4) $\sqrt{27}$

In 27–30, for each expression:

a. Use a calculator to find a rational approximation of the expression.

b. Write the original expression in simplest radical form.

c. Use a calculator to find a rational approximation of the answer to part **b**.

d. Are the approximations in parts **a** and **c** equal?

27. $\sqrt{300}$ **28.** $\sqrt{180}$ **29.** $2\sqrt{288}$ **30.** $\frac{1}{3}\sqrt{252}$

31. **a.** Does $\sqrt{9 + 16} = \sqrt{9} + \sqrt{16}$? Explain your answer.

b. Is finding a square root distributive over addition?

32. **a.** Does $\sqrt{169 - 25} = \sqrt{169} - \sqrt{25}$? Explain your answer.

b. Is finding a square root distributive over subtraction?

12-5 ADDITION AND SUBTRACTION OF RADICALS

Radicals are exact values. Computations sometimes involve many radicals, as in the following example:

$$\sqrt{12} + \sqrt{75} + 3\sqrt{3}$$

Rather than use approximations obtained with a calculator to find this sum, it may be important to express the answer as an *exact* value in radical form. To learn how to perform computations with radicals, we need to recall some algebraic concepts.

Adding and Subtracting Like Radicals

We have learned how to add like terms in algebra:

$$2x + 5x + 3x = 10x$$

If we replace the variable x by an irrational number, $\sqrt{3}$, the following statement must be true:

$$2\sqrt{3} + 5\sqrt{3} + 3\sqrt{3} = 10\sqrt{3}$$

Like radicals are radicals that have the same index and the same radicand. For example, $2\sqrt{3}$, $5\sqrt{3}$, and $3\sqrt{3}$ are like radicals and $6\sqrt[3]{7}$ and $\sqrt[3]{7}$ are like radicals.

To demonstrate that the sum of like radicals can be written as a single term, we can apply the concept used to add like terms, namely, the distributive property:

$$2\sqrt{3} + 5\sqrt{3} + 3\sqrt{3} = \sqrt{3}(2 + 5 + 3) = \sqrt{3}(10) = 10\sqrt{3}$$

Similarly,

$$6\sqrt[3]{7} - \sqrt[3]{7} = \sqrt[3]{7}(6 - 1) = \sqrt[3]{7}(5) = 5\sqrt[3]{7}$$

Procedure

To add or subtract like terms that contain like radicals:

1. Add or subtract the coefficients of the radicals.
2. Multiply the sum or difference obtained by the common radical.

Adding and Subtracting Unlike Radicals

Unlike radicals are radicals that have different radicands or different indexes, or both. For example:

- $3\sqrt{5}$ and $2\sqrt{2}$ are unlike radicals because their radicands are different.
- $\sqrt[3]{2}$ and $\sqrt{2}$ are unlike radicals because their indexes are different.
- $9\sqrt{10}$ and $\sqrt[3]{4}$ are unlike radicals because their radicands and indexes are different.

The sum or difference of unlike radicals cannot always be expressed as a single term. For instance:

- The sum of $3\sqrt{5}$ and $2\sqrt{2}$ is $3\sqrt{5} + 2\sqrt{2}$.
- The difference of $5\sqrt{7}$ and $\sqrt{11}$ is $5\sqrt{7} - \sqrt{11}$.

However, it is sometimes possible to transform unlike radicals into equivalent like radicals. These like radicals can then be combined into a single term. Let us return to the problem posed at the start of this section:

$$\begin{aligned}\sqrt{12} + \sqrt{75} + 3\sqrt{3} &= \sqrt{4 \cdot 3} + \sqrt{25 \cdot 3} + 3\sqrt{3}\\ &= 2\sqrt{3} + 5\sqrt{3} + 3\sqrt{3}\\ &= 10\sqrt{3}\end{aligned}$$

Procedure

To combine unlike radicals:

1. Simplify each radical if possible.
2. Combine like radicals by using the distributive property.
3. Write the indicated sum or difference of the unlike radicals.

EXAMPLE 1

Combine: **a.** $8\sqrt{5} + \sqrt{5} - 2\sqrt{5}$ **b.** $6\sqrt{n} - 2\sqrt{49n}$

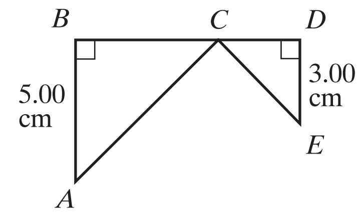
Solution **a.** $8\sqrt{5} + \sqrt{5} - 2\sqrt{5} = \sqrt{5}(8 + 1 - 2) = \sqrt{5}(7) = 7\sqrt{5}$

b. $6\sqrt{n} - 2\sqrt{49n} = 6\sqrt{n} - 2(7)\sqrt{n} = \sqrt{n}(6 - 14) = -8\sqrt{n}$

Answers **a.** $7\sqrt{5}$ **b.** $-8\sqrt{n}$

EXAMPLE 2

Alexis drew the figure at the right. Triangles ABC and CDE are isosceles right triangles. $AB = 5.00$ centimeters and $DE = 3.00$ centimeters.



a. Find AC and CE.

b. Find, in simplest radical form, $AC + CE$.

c. Alexis measured $AC + CE$ and found the measure to be 11.25 centimeters. Find the percent of error of the measurement that Alexis made. Express the answer to the nearest hundredth of a percent.

Solution **a.** Use the Pythagorean Theorem.

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ &= 5^2 + 5^2 \\ &= 25 + 25 \\ &= 50 \\ AC &= \sqrt{50} \end{aligned} \qquad\qquad \begin{aligned} CE^2 &= CD^2 + DE^2 \\ &= 3^2 + 3^2 \\ &= 9 + 9 \\ &= 18 \\ CE &= \sqrt{18} \end{aligned}$$

b. $AC + CE = \sqrt{50} + \sqrt{18} = \sqrt{25}\cdot\sqrt{2} + \sqrt{9}\cdot\sqrt{2} = 5\sqrt{2} + 3\sqrt{2} = 8\sqrt{2}$

c. Error $= 8\sqrt{2} - 11.25$

Percent of error $= \dfrac{8\sqrt{2} - 11.25}{8\sqrt{2}}$

ENTER: **(** 8 **2nd** **√** 2 **)** **−** 11.25 **)** **÷**

(8 **2nd** **√** 2 **)** **)** **ENTER**

DISPLAY:

```
(8√(2)-11.25)/(8
√(2))
            .005631089
```

Multiply the number in the display by 100 to change to percent.

Percent of error $= 0.5631089\% = 0.56\%$

Answers **a.** $AC = \sqrt{50}$ cm, $CE = \sqrt{18}$ cm **b.** $8\sqrt{2}$ cm **c.** 0.56%

EXERCISES

Writing About Mathematics

1. Compare adding fractions with adding radicals. How are the two operations alike and how are they different?

2. Marc said that $3\sqrt{5} - \sqrt{5} = 3$. Do you agree with Marc? Explain why or why not.

Developing Skills

In 3–23, in each case, combine the radicals. Assume that all variables represent positive numbers.

3. $8\sqrt{5} + \sqrt{5}$

4. $5\sqrt{3} + 2\sqrt{3} + 8\sqrt{3}$

5. $7\sqrt{2} - \sqrt{2}$

6. $4\sqrt{3} + 2\sqrt{3} - 6\sqrt{3}$

7. $5\sqrt{3} + \sqrt{3} - 2\sqrt{3}$

8. $4\sqrt{7} - \sqrt{7} - 5\sqrt{7}$

9. $3\sqrt{5} + 6\sqrt{2} - 3\sqrt{2} + \sqrt{5}$

10. $9\sqrt{x} + 3\sqrt{x}$

11. $15\sqrt{y} - 7\sqrt{y}$

12. $\sqrt{2} + \sqrt{50}$

13. $\sqrt{27} + \sqrt{75}$

14. $\sqrt{80} - \sqrt{5}$

15. $\sqrt{12} - \sqrt{48} + \sqrt{3}$

16. $\sqrt{0.98} - 4\sqrt{0.08} + 3\sqrt{1.28}$

17. $\sqrt{0.2} + \sqrt{0.45}$

18. $\sqrt{\frac{8}{9}} - \sqrt{72}$

19. $\sqrt{\frac{3}{4}} + \sqrt{\frac{1}{3}}$

20. $\sqrt{7a} + \sqrt{28a}$

21. $\sqrt{100b} - \sqrt{64b} + \sqrt{9b}$

22. $3\sqrt{3x} - \sqrt{12x}$

23. $\sqrt{3a^2} + \sqrt{12a^2}$

24. $x\sqrt{a^2} + 6\sqrt{a} - 3\sqrt{a}$

In 25–27, in each case, select the numeral preceding the correct choice.

25. The difference $5\sqrt{2} - \sqrt{32}$ is equivalent to

(1) $\sqrt{2}$ (2) $9\sqrt{2}$ (3) $4\sqrt{30}$ (4) $5\sqrt{30}$

26. The sum $3\sqrt{8} + 6\sqrt{2}$ is equivalent to

(1) $9\sqrt{10}$ (2) $\sqrt{72}$ (3) $18\sqrt{10}$ (4) $12\sqrt{2}$

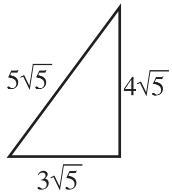
27. The sum of $\sqrt{12}$ and $\sqrt{27}$ is equivalent to

(1) $\sqrt{39}$ (2) $5\sqrt{6}$ (3) $13\sqrt{3}$ (4) $5\sqrt{3}$

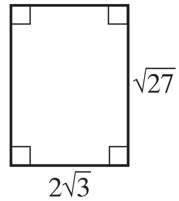
Applying Skills

In 28 and 29: **a.** Express the perimeter of the figure in simplest radical form. **b.** Using a calculator, approximate the expression obtained in part **a** to the *nearest thousandth*.

28.

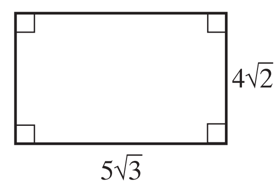


29.



30. On the way to softball practice, Maggie walks diagonally through a square field and a rectangular field. The square field has a length of 60 yards. The rectangular field has a length of 70 yards and a width of 10 yards. What is the total distance Maggie walks through the fields?

12-6 MULTIPLICATION OF SQUARE-ROOT RADICALS



To find the area of the rectangle pictured at the right, we multiply $5\sqrt{3}$ by $4\sqrt{2}$. We have learned that $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ when a and b are nonnegative numbers. To multiply $4\sqrt{2}$ by $5\sqrt{3}$, we use the commutative and associative laws of multiplication as follows:

$$\left(4\sqrt{2}\right)\left(5\sqrt{3}\right) = (4)(5)\left(\sqrt{2}\right)\left(\sqrt{3}\right) = (4 \cdot 5)\left(\sqrt{2 \cdot 3}\right) = 20\sqrt{6}$$

In general, if a and b are nonnegative numbers:

$$\left(x\sqrt{a}\right)\left(y\sqrt{b}\right) = xy\sqrt{ab}$$

Procedure

To multiply two monomial square roots:

1. Multiply the coefficients to find the coefficient of the product.
2. Multiply the radicands to find the radicand of the product.
3. If possible, simplify the result.

EXAMPLE 1

a. Multiply $\left(3\sqrt{6}\right)\left(5\sqrt{2}\right)$ and write the product in simplest radical form.

b. Check the work performed in part **a** using a calculator.

Solution **a.** $\left(3\sqrt{6}\right)\left(5\sqrt{2}\right) = 3(5)\sqrt{6(2)}$

$= 15\sqrt{12}$

$= 15\left(\sqrt{4}\right)\left(\sqrt{3}\right)$

$= 15(2)\sqrt{3}$

$= 30\sqrt{3}$ *Answer*

b. To check, evaluate $\left(3\sqrt{6}\right)\left(5\sqrt{2}\right)$.

ENTER: 3 **2nd** $\sqrt{\ }$ 6 **)** × 5 **2nd** $\sqrt{\ }$ 2 **)** **ENTER**

DISPLAY:
```
3√(6)*5√(2)
        51.96152423
```

Then evaluate $30\sqrt{3}$.

ENTER: 30 **2nd** $\sqrt{\ }$ 3 **ENTER**

DISPLAY:
```
30√(3
        51.96152423
```

Therefore, $\left(3\sqrt{6}\right)\left(5\sqrt{2}\right) = 30\sqrt{3}$ appears to be true.

EXAMPLE 2

Find the value of $\left(2\sqrt{3}\right)^2$.

Solution $\left(2\sqrt{3}\right)^2 = \left(2\sqrt{3}\right)\left(2\sqrt{3}\right) = 2(2)\left(\sqrt{3}\right)\left(\sqrt{3}\right) = 4\sqrt{9} = 4(3) = 12$

Alternative Solution $\left(2\sqrt{3}\right)^2 = (2)^2\left(\sqrt{3}\right)^2 = 4(3) = 12$

Answer 12

EXAMPLE 3

Find the indicated product: $\sqrt{3x} \cdot \sqrt{6x}$, $(x > 0)$.

Solution $\sqrt{3x} \cdot \sqrt{6x} = \sqrt{3x \cdot 6x} = \sqrt{18x^2} = \sqrt{9x^2} \cdot \sqrt{2} = 3x\sqrt{2}$ *Answer*

EXERCISES

Writing About Mathematics

1. When a and b are unequal prime numbers, is $\sqrt{ab}$ rational or irrational? Explain your answer.

2. Is $\sqrt{4a^2}$ a rational number for all values of a? Explain your answer.

Developing Skills

In 3–26: in each case multiply, or raise to the power, as indicated. Then simplify the result. Assume that all variables represent positive numbers.

3. $\sqrt{3} \cdot \sqrt{3}$ **4.** $\sqrt{7} \cdot \sqrt{7}$ **5.** $\sqrt{a} \cdot \sqrt{a}$

6. $\sqrt{2x} \cdot \sqrt{2x}$ **7.** $\sqrt{12} \cdot \sqrt{12}$ **8.** $8\sqrt{18} \cdot 3\sqrt{8}$

9. $\sqrt{14} \cdot \sqrt{2}$ **10.** $\sqrt{60} \cdot \sqrt{5}$ **11.** $3\sqrt{6} \cdot \sqrt{3}$

12. $5\sqrt{8} \cdot 7\sqrt{3}$ **13.** $\frac{2}{3}\sqrt{24} \cdot 9\sqrt{3}$ **14.** $5\sqrt{6} \cdot \frac{2}{3}\sqrt{15}$

15. $\left(-4\sqrt{a}\right)\left(3\sqrt{a}\right)$ **16.** $\left(-\frac{1}{2}\sqrt{y}\right)\left(-6\sqrt{y}\right)$ **17.** $\left(\sqrt{2}\right)^2$

18. $\left(\sqrt{y}\right)^2$ **19.** $\left(\sqrt{t}\right)^2$ **20.** $\left(3\sqrt{6}\right)^2$

21. $\left(\sqrt{25x}\right)\left(\sqrt{4x}\right)$ **22.** $\left(\sqrt{27a}\right)\left(\sqrt{3a}\right)$ **23.** $\left(\sqrt{15x}\right)\left(\sqrt{3x}\right)$

24. $\left(\sqrt{9a}\right)\left(\sqrt{ab}\right)$ **25.** $\left(\sqrt{5x}\right)^2$ **26.** $\left(2\sqrt{t}\right)^2$

In 27–33: **a.** Perform each indicated operation. **b.** State whether the product is an irrational number or a rational number.

27. $\left(5\sqrt{12}\right)\left(4\sqrt{3}\right)$ **28.** $\left(3\sqrt{2}\right)\left(2\sqrt{32}\right)$ **29.** $\left(4\sqrt{6}\right)\left(9\sqrt{3}\right)$ **30.** $\left(8\sqrt{5}\right)\left(\frac{1}{2}\sqrt{10}\right)$

31. $\left(\frac{2}{3}\sqrt{5}\right)^2$ **32.** $\left(\frac{1}{6}\sqrt{8}\right)\left(\frac{1}{2}\sqrt{18}\right)$ **33.** $\left(\frac{5}{7}\sqrt{7}\right)^3$ **34.** $\left(11\sqrt{38}\right)\left(\frac{1}{11}\sqrt{45}\right)$

35. Two square-root radicals that are *irrational numbers* are multiplied.

a. Give two examples where the product of these radicals is also an irrational number.

b. Give two examples where the product of these radicals is a rational number.

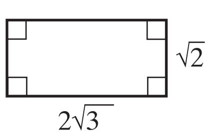
Applying Skills

In 36–39, in each case, find the area of the square in which the length of each side is given.

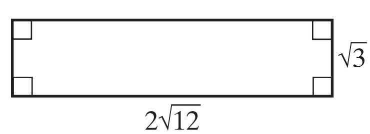
36. $\sqrt{2}$ **37.** $2\sqrt{3}$ **38.** $6\sqrt{2}$ **39.** $5\sqrt{3}$

In 40 and 41: **a.** Express the area of the figure in simplest radical form. **b.** Check the work performed in part **a** by using a calculator.

40.



41.



12-7 DIVISION OF SQUARE-ROOT RADICALS

Since	$\sqrt{\frac{4}{9}} = \frac{2}{3}$	Since	$\sqrt{\frac{16}{25}} = \frac{4}{5}$
and	$\frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$,	and	$\frac{\sqrt{16}}{\sqrt{25}} = \frac{4}{5}$,
then	$\sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}}$.	then	$\sqrt{\frac{16}{25}} = \frac{\sqrt{16}}{\sqrt{25}}$.

These examples illustrate the following property of square-root radicals:

► **The square root of a fraction that is the quotient of two positive numbers is equal to the square root of the numerator divided by the square root of the denominator.**

In general, if a and b are positive numbers:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad \text{and} \quad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

We use this principle to divide $\sqrt{72}$ by $\sqrt{8}$. In this example, notice that the quotient of two irrational numbers is a *rational* number.

$$\frac{\sqrt{72}}{\sqrt{8}} = \sqrt{\frac{72}{8}} = \sqrt{9} = 3$$

We can also divide radical terms by using the property of fractions:

$$\frac{ac}{bd} = \frac{a}{b} \cdot \frac{c}{d}$$

Note, in the following example, that the quotient of two irrational numbers is irrational:

$$\frac{6\sqrt{10}}{3\sqrt{2}} = \frac{6}{3} \cdot \frac{\sqrt{10}}{\sqrt{2}} = \frac{6}{3} \cdot \sqrt{\frac{10}{2}} = 2\sqrt{5}$$

In general, if a and b are positive, and $y \neq 0$:

$$\frac{x\sqrt{a}}{y\sqrt{b}} = \frac{x}{y}\sqrt{\frac{a}{b}}$$

Procedure

To divide two monomial square roots:

1. Divide the coefficients to find the coefficient of the quotient.
2. Divide the radicands to find the radicand of the quotient.
3. If possible, simplify the result.

EXAMPLE 1

Divide $8\sqrt{48}$ by $4\sqrt{2}$, and simplify the quotient.

Solution $8\sqrt{48} \div 4\sqrt{2} = \frac{8}{4}\sqrt{\frac{48}{2}} = 2\sqrt{24} = 2\left(\sqrt{4}\right)\left(\sqrt{6}\right) = 2(2)\left(\sqrt{6}\right) = 4\sqrt{6}$ *Answer*

EXAMPLE 2

Find the indicated division: $\frac{\sqrt{2x^2y^3z}}{\sqrt{6y}}$, $(x > 0, y > 0, z > 0)$.

Solution $\frac{\sqrt{2x^2y^3z}}{\sqrt{6y}} = \sqrt{\frac{1}{3}} \cdot \sqrt{x^2y^2z} = \sqrt{\frac{3}{9}} \cdot \sqrt{x^2y^2z} = \sqrt{\frac{1}{9}x^2y^2} \cdot \sqrt{3z} = \frac{1}{3}xy\sqrt{3z}$ *Answer*

EXERCISES

Writing About Mathematics

1. Ross simplified $\sqrt{\frac{16}{81}}$ by writing $\frac{\sqrt{16}}{\sqrt{81}} = \frac{4}{9} = \frac{2}{3}$. Do you agree with Ross? Explain why or why not.

2. Is $\sqrt{\frac{1}{16}}$ rational or irrational? Explain your answer.

Developing Skills

In 3–18, divide. Write the quotient in simplest form. Assume that all variables represent positive numbers.

3. $\sqrt{72} \div \sqrt{2}$
4. $\sqrt{75} \div \sqrt{3}$
5. $\sqrt{70} \div \sqrt{10}$
6. $\sqrt{14} \div \sqrt{2}$
7. $8\sqrt{48} \div 2\sqrt{3}$
8. $\sqrt{24} \div \sqrt{2}$
9. $\sqrt{150} \div \sqrt{3}$
10. $21\sqrt{40} \div \sqrt{5}$
11. $9\sqrt{6} \div 3\sqrt{6}$
12. $7\sqrt{3} \div 3\sqrt{3}$
13. $2\sqrt{2} \div 8\sqrt{2}$
14. $\sqrt{9y} \div \sqrt{y}$
15. $\frac{12\sqrt{20}}{3\sqrt{5}}$
16. $\frac{20\sqrt{50}}{4\sqrt{2}}$
17. $\frac{a\sqrt{b^3c^4}}{\sqrt{a^2}}$
18. $\frac{3\sqrt{x^3y}}{6\sqrt{z}}$

In 19–26, state whether each quotient is a rational number or an irrational number.

19. $\frac{\sqrt{5}}{7}$
20. $\frac{\sqrt{50}}{\sqrt{2}}$
21. $\frac{\sqrt{18}}{\sqrt{3}}$
22. $\frac{\sqrt{49}}{\sqrt{7}}$
23. $\frac{\sqrt{9}}{\sqrt{16}}$
24. $\frac{\sqrt{18}}{\sqrt{25}}$
25. $\frac{25\sqrt{24}}{5\sqrt{2}}$
26. $\frac{3\sqrt{54}}{6\sqrt{3}}$

In 27–34, simplify each expression. Assume that all variables represent positive numbers.

27. $\sqrt{\frac{36}{49}}$ **28.** $\sqrt{\frac{3}{4}}$ **29.** $4\sqrt{\frac{5}{16}}$ **30.** $\sqrt{\frac{8}{49}}$

31. $10\sqrt{\frac{8}{25}}$ **32.** $\frac{9}{18}\sqrt{\frac{xy^2}{y}}$ **33.** $\sqrt{\frac{a^6b^5c^4}{a^4b^3c^2}}$ **34.** $\frac{15}{3}\sqrt{\frac{a^5}{a^2b^2}}$

Applying Skills

In 35–38, in each case, the area A of a parallelogram and the measure of its base b are given. Find the height h of the parallelogram, expressed in simplest form.

35. $A = 7\sqrt{12}, b = 7\sqrt{3}$ **36.** $A = \sqrt{640}, b = \sqrt{32}$

37. $A = 8\sqrt{45}, b = 2\sqrt{15}$ **38.** $A = 2\sqrt{98}, b = \sqrt{32}$

CHAPTER SUMMARY

A **radical**, which is the root of a quantity, is written in its general form as $\sqrt[n]{b}$. A radical consists of a **radicand**, b, placed under a **radical sign**, $\sqrt{\ }$, with an **index**, n.

Finding the square root of a quantity reverses the result of the operation of squaring. A **square-root radical** has an index of 2, which is generally not written. Thus, $\sqrt[2]{49} = \sqrt{49} = 7$ because $7^2 = 49$. In general, for nonnegative numbers b and x, $\sqrt{b} = x$ if and only if $x^2 = b$.

If k is a positive number that is a perfect square, then $\sqrt{k}$ is a rational number. If k is positive but not a perfect square, then $\sqrt{k}$ is an irrational number.

Every positive number has two square roots: a positive root called the **principal square root**, and a negative root. These roots have the same absolute value:

Principal Square Root	*Negative Square Root*	*Both Square Roots*
$\sqrt{x^2} = \lvert x \rvert$	$-\sqrt{x^2} = -\lvert x \rvert$	$\pm\sqrt{x^2} = \pm x$

Finding the **cube root** of a number is the inverse of the operation of cubing. Thus, $\sqrt[3]{64} = 4$ because $4^3 = 4 \times 4 \times 4 = 64$. In general, $\sqrt[3]{b} = x$ if and only if $x^3 = b$.

Like radicals have the same radicand and the same index. For example, $2\sqrt{7}$ and $3\sqrt{7}$. **Unlike radicals** can differ in their radicands ($\sqrt{7}$ and $\sqrt{2}$), in their indexes ($\sqrt{7}$ and $\sqrt[3]{7}$), or in both ($\sqrt[4]{7}$ and $\sqrt[3]{6}$).

A square root of a positive integer is **simplified** by factoring out the square root of its greatest perfect square. The radicand of a simplified radical, then, has no perfect square factor other than 1. When a radical is irrational, the radical expresses its *exact* value. Most calculators display only **rational approximations** of radicals that are irrational numbers.

Operations with radicals include:

1. **Addition and subtraction:** Combine like radicals by adding or subtracting their coefficients and then multiplying this result by their common radical. The sum or difference of unlike radicals, unless transformed to equivalent like radicals, cannot be expressed as a single term.

2. **Multiplication and division:** For all radicals whose denominators are not equal to 0, multiply (or divide) coefficients, multiply (or divide) radicands, and simplify. The general rules for these operations are as follows:

$$\left(x\sqrt{a}\right)\left(y\sqrt{b}\right) = xy\sqrt{ab} \quad \text{and} \quad \frac{x\sqrt{a}}{y\sqrt{b}} = \frac{x}{y}\sqrt{\frac{a}{b}}$$

VOCABULARY

12-1 Radicand • Radical • Perfect square • Principal square root • Cube root • Index

12-4 Simplest form of a square-root radical

12-5 Like radicals • Unlike radicals

REVIEW EXERCISES

1. When $\sqrt{\frac{a}{b}}$ is an integer, what is the relationship between a and b?
2. When a is a positive perfect square and b is a positive number that is not a perfect square, is $\sqrt{ab}$ rational or irrational? Explain your answer.
3. Write the principal square root of 1,225.
4. Write the following numbers in order starting with the smallest: $\sqrt{18}$, $2\sqrt{2}$, 3.

In 5–14, write each number in simplest form.

5. $\sqrt{\frac{9}{25}}$ **6.** $-\sqrt{49}$ **7.** $\sqrt[3]{-27}$ **8.** $\pm\sqrt{1.21}$

9. $\sqrt{400y^4}$ **10.** $\sqrt{180}$ **11.** $3\sqrt{18}$ **12.** $\frac{1}{2}\sqrt{28}$

13. $\sqrt{48b^3}$, $b > 0$ **14.** $\sqrt{0.01m^{16}}$ **15.** $\sqrt{\frac{9}{27}x^3y^5}$, $x > 0$, $y > 0$ **16.** $\sqrt{0.25a^8b^{10}}$

In 17–20, in each case, solve for the variable, using the set of real numbers as the replacement set.

17. $y^2 = 81$ **18.** $m^2 = 0.09$ **19.** $3x^2 = 600$ **20.** $2k^2 - 144 = 0$

21. **a.** Use a calculator to evaluate $\sqrt{315.4176}$.

b. Is $\sqrt{315.4176}$ a rational or an irrational number? Explain your answer.

In 22–33, in each case, perform the indicated operation and simplify the result.

22. $\sqrt{18} + \sqrt{8} - \sqrt{32}$

23. $3\sqrt{20} - 2\sqrt{45}$

24. $2\sqrt{50} - \sqrt{98} + \frac{1}{2}\sqrt{72}$

25. $\sqrt{75} - 3\sqrt{12}$

26. $8\sqrt{2}\left(2\sqrt{2}\right)$

27. $\left(3\sqrt{5}\right)^2$

28. $2\sqrt{7}\left(\sqrt{70}\right)$

29. $\sqrt{98} \div \sqrt{2}$

30. $\frac{16\sqrt{21}}{2\sqrt{7}}$

31. $\frac{5\sqrt{162}}{9\sqrt{50}}$

32. $\sqrt{3}\left(\sqrt{24}\right) - \sqrt{5}\left(\sqrt{10}\right)$

33. $\frac{\sqrt{80}}{\sqrt{2}} + \sqrt{6}\left(\sqrt{60}\right)$

In 34 and 35, in each case, select the numeral preceding the correct choice.

34. The expression $\sqrt{108} - \sqrt{3}$ is equivalent to

(1) $\sqrt{105}$ (2) $35\sqrt{3}$ (3) $5\sqrt{3}$ (4) 6

35. The sum of $9\sqrt{2}$ and $\sqrt{32}$ is

(1) $9\sqrt{34}$ (2) $13\sqrt{2}$ (3) $10\sqrt{34}$ (4) 15

In 36–39, for each irrational number given, write a rational approximation: **a.** as shown on a calculator display **b.** rounded to four significant digits.

36. $\sqrt{194}$ **37.** $\sqrt[3]{16}$ **38.** $-\sqrt{0.7}$ **39.** $\sqrt[5]{-27}$

40. The area of a square is 28.00 square meters.

a. Find, to the *nearest thousandth* of a meter, the length of one side of the square.

b. Find, to the *nearest thousandth* of a meter, the perimeter of the square.

c. Explain why the answer to part **b** is *not* equal to 4 times the answer to part **a**.

41. Write as a polynomial in simplest form: $\left(2x + \sqrt{3}\right)\left(x - \sqrt{3}\right)$.

42. What is the product of $-3.5x^2$ and $-6.2x^3$?

Exploration

STEP 1. On a sheet of graph paper, draw the positive ray of the real number line. Draw a square, using the interval from 0 to 1 as one side of the square. Draw the diagonal of this square from 0 to its opposite vertex.

The length of the diagonal is $\sqrt{2}$. Why? Place the point of a compass at 0 and the pencil of a compass at the opposite vertex of the square so that the measure of the opening of the pair of compasses is $\sqrt{2}$. Keep the point of the compass at 0 and use the pencil to mark the endpoint of a segment of length $\sqrt{2}$ on the number line.

STEP 2. Using the same number line, draw a rectangle whose dimensions are $\sqrt{2}$ by 1, using the interval on the number line from 0 to $\sqrt{2}$ as one side. Draw the diagonal of this rectangle from 0 to the opposite vertex. The length of the diagonal is $\sqrt{3}$. Place the point of a pair of compasses at 0 and the pencil at the opposite vertex of the rectangle so that the measure of the opening of the pair of compasses is $\sqrt{3}$. Keep the point of the compasses at 0 and use the pencil to mark the endpoint of a segment of length $\sqrt{3}$ on the number line.

STEP 3. Repeat step 2, drawing a rectangle whose dimensions are $\sqrt{3}$ by 1 to locate $\sqrt{4}$ on the number line. This point should coincide with 2 on the number line.

STEP 4. Explain how these steps can be used to locate $\sqrt{n}$ for any positive integer n.

CUMULATIVE REVIEW — CHAPTERS 1–12

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. A flagpole casts a shadow five feet long at the same time that a man who is six feet tall casts a shadow that is two feet long. How tall is the flagpole?

(1) 12 feet (2) 15 feet (3) 18 feet (4) 24 feet

2. Which of the following is an irrational number?

(1) $\sqrt{12}\left(\sqrt{3}\right)$ (2) $\sqrt{24} \div \sqrt{6}$ (3) $\sqrt{5} - \sqrt{5}$ (4) $3\sqrt{5} - \sqrt{5}$

3. Parallelogram $ABCD$ is drawn on the coordinate plane with the vertices $A(0, 0)$, $B(8, 0)$, $C(10, 5)$, and $D(2, 5)$. The number of square units in the area of $ABCD$ is

(1) 40 (2) 20 (3) 16 (4) 10

4. The product $(3a - 2)(2a + 3)$ can be written as

(1) $6a^2 - 6$
(2) $6a^2 + a - 6$
(3) $6a^2 - 5a - 6$
(4) $6a^2 + 5a - 6$

5. If $0.2x - 8 = x + 4$, then x equals
(1) 120 (2) 12 (3) -15 (4) 15

6. If the height of a right circular cylinder is 12 centimeters and the measure of the diameter of a base is 8 centimeters, then the volume of the cylinder is
(1) 768π (2) 192π (3) 96π (4) 48π

7. The identity $3(a + 7) = 3a + 21$ is an example of
(1) the additive inverse property
(2) the associative property for addition
(3) the commutative property for addition
(4) the distributive property of multiplication over addition

8. The value of a share of stock decreased from \$24.50 to \$22.05. The percent of decrease was
(1) 1% (2) 10% (3) 11% (4) 90%

9. When written in scientific notation, 384.5 is equal to
(1) 38.45×10^1 (2) 3.845×10^2 (3) 3.845×10^3 (4) 3.845×10^{-2}

10. In right triangle ABC, $\angle C$ is the right angle. The cosine of $\angle B$ is
(1) $\frac{AC}{AB}$ (2) $\frac{BC}{AC}$ (3) $\frac{BC}{AB}$ (4) $\frac{AC}{BC}$

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. A car uses $\frac{3}{4}$ of a tank of gasoline to travel 600 kilometers. The tank holds 48 liters of gasoline. How far can the car go on one liter of gasoline?

12. A ramp that is 20.0 feet long makes an angle of 12.5° with the ground. What is the perpendicular distance from the top of the ramp to the ground?

Part III

Answer all questions in this part. Each correct answer will receive 3 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. $ABCD$ is a trapezoid with $\overline{BC} \perp \overline{AB}$ and $\overline{BC} \perp \overline{CD}$, $AB = 13$, $BC = 12$, and $CD = 8$. A line segment is drawn from A to E, the midpoint of $\overline{CD}$.

a. Find the area of $\triangle AED$.

b. Find the perimeter of $\triangle AED$.

14. Maria's garden is in the shape of a rectangle that is twice as long as it is wide. Maria increases the width by 2 feet, making the garden 1.5 times as long as it is wide. What are the dimensions of the original garden?

Part IV

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. The length of the base of an isosceles triangle is $10\sqrt{2}$ and the length of the altitude is $12\sqrt{2}$. Express the perimeter as an exact value in simplest form.

16. In the coordinate plane, O is the origin, A is a point on the y-axis, and B is a point on the x-axis. The slope of $\overleftrightarrow{AB}$ is -2 and its y-intercept is 8.

a. Write the equation of $\overleftrightarrow{AB}$.

b. Draw $\overleftrightarrow{AB}$ on graph paper.

c. What are the coordinates of B?

d. What is the x-intercept of $\overleftrightarrow{AB}$?